# Hybrid quantum computing

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## Abstract

Necessary and sufficient conditions are given for the construction of a hybrid quantum computer that operates on both continuous and discrete quantum variables. Such hybrid computers are shown to be more efficient than conventional quantum computers for performing a variety of quantum algorithms, such as computing eigenvectors and eigenvalues.

03.65.Bz, 05.30.-d, 89.70.+c

Quantum computers are devices that process information in a way that preserves quantum coherence [1–6,8], [9–11] The most common model of quantum computation deals with coherent logical operations on two-state quantum variables known as qubits. Quantum computation can also be performed on variables with three or more states, and is well-defined even when the underlying degrees of freedom are continuous [12–15]. This paper investigates hybrid quantum computers that operate on both discrete and continuous quantum variables. It is shown that a simple set of operations (hybrid quantum logic gates) can be used to approximate arbitrary tranformations of the variables. Hybrid versions of quantum algorithms are discussed and a hybrid version of an algorithm for finding eigenvalues and eigenvectors is presented. Hybrid quantum algorithms can have a number of advantages over conventional quantum algorithms, including lower computational complexity and an enhanced resistance to noise and decoherence.

The primary reason for investigating hybrid quantum computers is that nature contains both discrete quantum variables such as nuclear spins, photon polarizations, and atomic energy levels, and continuous variables such as position, momentum, and the quadrature amplitudes of the electromagnetic field. In conventional quantum computation, continuous variables are something of a nuisance: either they figure as sources of noise and decoherence, as in the case of environmental baths of harmonic oscillators, or they must be restricted to a discrete set of states by cooling, as in the case of the oscillatory modes of ions in ion-trap quantum computers. In hybrid quantum computation, by contrast, the full range of continuous quantum variables can be put to use.

The basic model for performing quantum computation using a hybrid of continuous and discrete variables follows the normal model for performing quantum computation using discrete or continuous variables on their own [7,15]. Assume that one has the ability to 'turn on' and 'turn off' the members of a set Hamiltonian operators $\{\pm H_j\}$, corresponding to the ability to apply unitary transformations of the form $e^{\pm iH_j t}$. The set of transformations that can be constructed in this fashion is the set of transformations of the form $e^{-iHt}$ where $H$ is a member of the algebra generated from the $H_j$ via commutation: i.e., since

$e^{iH_2t}e^{iH_1t}e^{-iH_2t}e^{-iH_1t} = e^{-[H_1,H_2]t^2} + O(t^3)$, the ability to turn on and turn off $\pm H_1$ and $\pm H_2$ allows one effectively to turn on and off $H = \pm i[H_1, H_2]$, etc. Transformations of the form $e^{-iHt}$ for non-inifinitesimal $t$ can then be built up from infinitesimal transformations to any desired degree of accuracy.

For the sake of ease of exposition, concentrate here on discrete variables (qubits) that are spins, characterized by the usual Pauli operators $\sigma_x, \sigma_y, \sigma_z$ and to continuous variables (qunats) that are harmonic oscillators characterized by the usual annihilation and creation operators $a, a^\dagger$ ($[a, a^\dagger] = 1$), and by the 'position' and 'momentum' operators $X = (a+a^\dagger)/2$, $P = (a - a^\dagger)/2i$, ($[X, P] = i$). It is convenient to think of the harmonic oscillators as modes of the electromagnetic field with $X$ and $P$ proportional to the quadrature amplitudes of the mode. The generalization to discrete variables with more than two states and to other forms of continuous variable is straightforward and will be discussed below.

To perform quantum computations one must be able to prepare one's variables in a desired state, perform quantum logic operations, and read out the results. Assume that it is possible to prepare the discrete variables in the state $|0\rangle \equiv |\uparrow\rangle_z$, and the continuous variables in the vacuum state $|0\rangle$: $a|0\rangle = 0$. Assume that it is possible to measure $\sigma_z$ for the discrete variables and $X$ for the continuous variables.

Now look at performing transformations of the variables. Begin with just a pair — one spin and one oscillator. Suppose that one can turn on and turn off the Hamiltonians

$$\{\pm\sigma_x X, \pm\sigma_z X, \pm\sigma_z P\}\,, \tag{1}$$

As will now be seen, the ability to turn on and off Hamiltonians from this set allows one to enact Hamiltonians that are arbitrary polynomials of the $\sigma$'s, $X$ and $P$. Note that these Hamiltonians all represent interactions between qubits and oscillators: this is physically realistic in the sense that transformations on physical spins or atoms are accomplished by making the spins interact with the electromagnetic field, and *vice versa*. In physically realizable situations, such as the ion traps and optical cavities discussed below, the interactions in 1 are turned on and off by applying laser or microwave pulses to couple discrete to continuous

degrees of freedom.

Now investigate what can be accomplished by turning on and off these interactions. If the spin is prepared in the state $|0\rangle$, then turning on the Hamiltonian $\sigma_z P$ is equivalent to turning on the Hamiltonian $P$ for the oscillator on its own. The Hamiltonian $X$ can be turned on in a similar fashion. In order to apply this Hamiltonian for a finite amount of time, the spin must be constantly reprepared in the state $|0\rangle$ or new spins in this state must be supplied. This operation allows the construction of coherent states of the oscillator.

Now start constructing effective Hamiltonians by the method of commutation above. Since $i[P, \sigma_x X] = \sigma_x$, we can effectively turn on the Hamiltonian $\sigma_x$. Similarly for the Hamiltonian $\pm i[P, \sigma_z X] = \pm\sigma_z$. And since $i[\sigma_z, \sigma_x] = 2\sigma_y$, any single qubit transformation $e^{-i\sigma t} \in SU(2)$ can be enacted by turning on and off Hamiltonians in the set. Since $i[\sigma_z P, \sigma_z X] = 2$, an arbitrary overall phase can also be turned on and off. That is, we can enact arbitrary single qubit transformations.

Now systematically build up higher order transformations. Since $i[\sigma_z X, \sigma_x X] = 2\sigma_y X^2$, and $i[\sigma_y X^2, \sigma_x X] = 2\sigma_z X^3$, etc., we can effectively turn on and off Hamiltonians of the form $\sigma X^n$, for arbitrary $\sigma$, $n$. Similarly, we can turn on and off Hamiltonians of the form $\sigma P^n$. By preparing the spin in the state $|0\rangle$ and turning on and off the Hamiltonians $\sigma_z X^m$, $\sigma_z P^n$, we can enact single oscillator transformations corresponding to Hamiltonians that are arbitrary Hermitian polynomials in $X$ and $P$. (Not all such Hamiltonians are bounded. Nonetheless, one can build up infinitesimal versions of such Hamiltonians and apply them for finite time to states for which they are bounded.)

So the simple set of Hamiltonians above allows the construction of arbitrary single qubit transformations and arbitrary polynomial transformations of the continuous variable, along with arbitrary interactions between the spin and the oscillator. Let us now look at more than one spin and one oscillator.

Since $i[\sigma_z^1 P, \sigma_z^2 X] = \sigma_z^1 \sigma_z^2$, we can turn on the interaction Hamiltonian $\sigma_z^1 \sigma_z^2$ between two spins 1 and 2 by making them both interact with the same oscillator. But the ability to turn on this Hamiltonian together with the ability to turn on arbitrary single-spin Hamiltonian

translates into the ability to perform arbitrary transformations on sets of spins: that is, one can perform arbitrary quantum logic operations on the qubits alone.

Similarly, since $i[\sigma_y X_1, \sigma_x X_2] = 2\sigma_z X_1 X_2$, the ability to make two oscillators interact with the same spin, initially in the state $|0\rangle$, allows one to turn on the Hamiltonian $X_1 X_2$ between the two oscillators 1 and 2. But this ability, together with the ability to turn on single oscillator Hamiltonians that are arbitrary Hermitian polynomials in $X$ and $P$, translates into the ability to turn on Hamiltonians that are arbitrary Hermitian polynomials of $X_i, P_i$ for all the oscillators together. So one can perform universal quantum computation on the continuous variables on their own.

Continuing with constructing Hamiltonians via commutation, the ability to prepare the $|0\rangle$ states for spins and oscillators, together with the ability to turn on and off the simple set 1 of Hamiltonians given above, allows one to effectively turn on and off Hamiltonians that are arbitrary Hermitian polynomials in $1, \sigma_x^j, \sigma_y^j, \sigma_z^j, X_k^m, P_k^n$. That is, one can perform universal quantum computation on the hybrid quantum computer.

How might such a hybrid quantum computer be realized? As it turns out, many existing designs for quantum computers are easily modified to perform hybrid quantum computation. For example, ion trap quantum computers [8,10] operate by coupling together the internal states of ions in an ion trap (qubits) via their motional state (harmonic oscillators). Existing schemes for performing quantum computation using ion traps only use the ground and first excited state of the oscillator corresponding to the fundamental mode of the ions in the trap, effectively treating the oscillator as a qubit. But the same methods that are used to couple the ions to the oscillator can just as well be used to apply the Hamiltonians in the set 1 above. An ion trap with many ions has many modes, each of which can be used as a continuous variable in the hybrid quantum computation. Similarly, the Pellizzari scheme for coupling together trapped atoms (qubits) via a cavity mode of the electromagnetic field can readily be altered to use the quadrature amplitudes of the modes of the cavity, rather than simply using the lowest two energy eigenstates of a mode as a qubit [11]. Other potential continuous variables that might be used for hybrid quantum computation are the translational states

of atoms in a Bose condensate, the continuum states of electrons in semiconductors, or the state of a Josephson junction circuit. Essentially any hybrid system that affords precise control over the interactions between discrete and continuous variables is a good candidate for a hybrid quantum computer.

An important concern in the construction of hybrid quantum computers is the problem of noise and decoherence. At first it might seem that continuous variables are likely to be more susceptible to noise than discrete variables. It is indeed true that more things can go wrong with a continuous variable than with a discrete variable. However, quantum error correction routines for continuous variables have been developed and require no greater overhead than those for discrete variables [12–14,16]. Although these routines are not yet technologically practical on existing devices, it may well be that improved versions of these routines combined with existing discrete quantum error correction routines will allow efficient quantum error correction for hybrid devices. In addition, as noted above, hybrid devices have the advantage that they include in the computation states and degrees of freedom that would normally be sources of noise, decoherence, and loss.

Now turn to applications of hybrid quantum computers. Where does the ability to perform manipulations of continuous variables as well as qubits give an advantage? The first point to note in constructing hybrid algorithms is that we must be careful to assume physically reasonable uses of hybrid variables—i.e., uses that do not require infinite or exponentially high precision. Even in the classical case, the use of continuous variables can give remarkable computational speed ups (the ability to solve NP-complete problems in polynomial time, the ability to find the the answer to uncomputable problems in finite time, etc.) if one allows arbitrary precision in manipulating and measuring continuous variables. By giving an explicit construction of the operations that can be used to perform continuous variable and hybrid quantum computation, however, we have implicitly avoided the use of infinite or excessive precision: all such operations would require infinite or excessive computational resources to construct, manipulate, and measure the desired over-precise states.

With this caveat in mind, turn to the operations that are relatively easy to perform us-

ing continuous quantum variables. A particularly useful subroutine in a variety of quantum algorithms is the quantum Fourier transform: $|x\rangle \rightarrow \sum_{y=1}^{q} e^{ixy}|y\rangle$. In the case of discrete quantum variables the quantum Fourier transform on $N$ qubits takes on the order of $N$ quantum logic operations to perform. Although this is an efficient algorithm it is nonetheless difficult at present to perform quantum Fourier transforms on more than a few qubits (the current record is three) [17]. By contrast, in the case of the continuous quantum variables $X$ and $P$, the quantum Fourier transform is trivial. If the eigenstates of $X$ with eigenvalue $x$ are written $|x\rangle$, then the eigenstates of $P$ with eigenvalue $p$ can be written $|p\rangle = (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} e^{ipx}|x\rangle dx$. That is, the eigenstates of $P$ are the quantum Fourier transform of the eigenstates of $X$. Applying the Hamiltonian $X^2 + P^2$ for a period of time $\pi/2$ takes $X \rightarrow P$ and performs the Fourier transform. The quantum Fourier transform on a continuous variable is accomplished by a single-step operation. The ease of performing the quantum Fourier transform on continuous variables suggests that in devising algorithms for hybrid quantum computers we look for problems in which the quantum Fourier transform plays a central role.

Perhaps the best known quantum algorithm in which the quantum Fourier transform plays a central role is Shor's algorithm for factoring large numbers [4]. Setting aside the difficulty of performing the other operations in this algorithm (such as modular exponentiation), it is immediately clear that using a continuous variable as the register on which to perform the quantum Fourier transform in Shor's algorithm would require an exponentially high precision in the preparation and manipulation of the continuous variable. (Hybrid quantum computation might still be used to speed up some aspects of Shor's algorithm; this possibility will be investigated elsewhere.)

A second problem in which the quantum Fourier transform plays a key role is that of simulating the dynamics of quantum systems [1,18–21]. Comparison with [18] shows that the ability of hybrid quantum computers to turn on and off simple Hamiltonians involving a few discrete and a few continuous variables at a time translates into the ability to perform efficient quantum simulations of hybrid systems.

A particularly valuable type of quantum simulation is one that allows the computation of spectra: using methods developed in [22,23,20] Abrams and Lloyd have developed algorithms for computing eigenvalues and eigenvectors of quantum systems and for obtaining improved estimates of the ground state [21]. In its original discrete form, the algorithm is somewhat involved. However, the fact that quantum Fourier transforms are straightforward to perform on continuous variables makes the Abrams-Lloyd algorithm particularly simple in the case of hybrid quantum computation. Here we show how to perform a quantum computation that computes the eigenvectors of a hybrid system and that writes the eigenvalues of the system onto a register consisting of a single continuous variable. The algorithm is a hybrid version of the discrete algorithms proposed in [22,23,20,21] and is closest in form to the discrete algorithm proposed in [20] for simulating von Neumann measurements on a quantum computer. Independently, Travaglione and Milburn [24] have shown how methods of hybrid quantum computation can be used to compute the eigenvectors of a continuous system and write the eigenvalues onto a discrete register.

First, prepare a single continuous variable such as a mode of the electromagnetic field in the squeezed state $|x=0\rangle = (1/\sqrt{2\pi}) \int_{-\infty}^{\infty} |p\rangle dp$. In any practical experiment, of course, such perfectly squeezed states are unavailable. Imperfectly squeezed or unsqueezed states will also work, however. As discussed below, the effect of imperfect squeezing is to decrease the resolution to which the spectrum can be obtained. Prepare a second system in the state $|\psi\rangle$ whose decomposition into energy eigenstates $|\psi\rangle = \sum_i \psi_i |E_i\rangle$ one wishes to obtain. Here we assume that the system is discrete; in general, however, system may be continuous, discrete, or a hybrid of continuous and discrete variables.

Next, using the methods of hybrid quantum computation described above, couple the system to the continuous variable via the coupling Hamiltonian $HP$, where $H$ is the Hamiltonian whose eigenvalues and eigenvectors are to be obtained. For $H$ to be efficiently simulatable, it must be equal to $\sum_k H_k$, where each $H_k$ acts on only a few variables at a time. Since $HP = \sum_k H_k P$, if $H$ is efficiently simulatable, so is $HP$, by the methods of hybrid

quantum computation described above. Writing $H = \sum_j E_j |E_j\rangle\langle E_j|$, the time evolution of the state of the system and the continuous variable is

$$\begin{aligned} & |\psi\rangle|x=0\rangle \\ \rightarrow \quad & e^{-iHPt}|\psi\rangle|x=0\rangle \\ = \; & e^{-i\sum_j E_j|E_j\rangle\langle E_j|Pt} \sum_j \psi_j |E_j\rangle|x=0\rangle \\ = \; & \sum_j e^{-iE_j tP}\psi_j|E_j\rangle|x=0\rangle \\ = \; & \sum_j \psi_i |E_j\rangle|x=E_j t\rangle\,, \end{aligned} \tag{2}$$

since $e^{-iPt}|x\rangle = |x+t\rangle$. Clearly, at this point, a measurement of the variable $X$ on the continuous variable will yield the result $x = tE_i$ with probability $\psi_i$, leaving the system in the state $|E_i = x/t\rangle$. That is, one can sample the spectral decomposition of $|\psi\rangle$, obtaining the eigenvalues $E_i$ together with their corresponding weights $|\psi_i|^2$ and eigenvectors $|E_i\rangle$. The process is highly efficient, requiring only the ability to prepare the initial squeezed state $|x=0\rangle$ and to apply the Hamiltonian $HP$.

The hybrid eigenvalue and eigenvector finding algorithm using a continuous variable to register the eigenvalue is more efficient than the corresponding algorithm using qubits to register the eigenvalue. Since the quantum Fourier transform is performed implicitly in the continuous register, fewer steps are required in the hybrid algorithm. In addition, unlike the conventional version of the algorithm, the hybrid version is insensitive to approximate decoherence of the register in the course of the computation: measuring the value $x$ of the register in the course of the coupling does not affect the ability of the algorithm to find eigenvectors and eigenvalues.

The requirement that the initial state of the continuous variable be perfectly squeezed can also be relaxed. Suppose that the initial state is in a Gaussian state $\int e^{-\beta x^2/2}|x\rangle dx$. For example, $\beta = 1$ gives the unsqueezed $n = 0$ vacuum state, while $\beta > 1$ gives partial squeezing in $X$. With this initial state for the continuous variable, after the algorithm has been run, the continuous variable and the system are in the state

$$\sum_j \int e^{-\beta x^2/2} |E_j\rangle |x + E_j t\rangle dx \, . \tag{3}$$

That is, the eigenvalues and eigenvectors are resolved to within an accuracy $1/t\sqrt{\beta}$. By coupling the system to the continuous variable for a sufficiently long time, the eigenvectors and eigenvalues of $H$ may be determined to an arbitrary degree of accuracy, even when the initial state is unsqueezed. Note that resolving the eigenvalues of a system with an exponentially large number of states requires exponential squeezing of the pointer state. But as noted in [21], this algorithm still provides a potentially exponential speedup over classical algorithms even when the eigenvalues are not determined to an exponential degree of accuracy.

Hybrid quantum computers are devices that perform quantum computations using both discrete variables such as spins and continuous variable such as position and momentum, or the quadrature amplitudes of the electromagnetic field. Hybrid quantum computation represents a natural extension of quantum computation using quantum bits alone: as the example of finding eigenvalues and eigenvectors presented here shows, hybrid quantum computations can be more efficient and less sensitive to noise and decoherence than conventional quantum computations. Nature supplies us with both discrete and continuous quantum variables: it is advantageous to use them.

This work was supported by DARPA/ARO under the QUIC initiative.

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